Robinson Instability under Cavity Voltage Feed-back

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1 Introduction

Robinson instabilities in storage rings are driven by a response of acceleration cavity voltage to synchrotron motion of a beam. Feed-back loops that control amplitude and phase of cavity voltage modify their response through beam-loading. We analyzed the linearized model of this system and obtained growth-rate and frequency shift of the instability under such feedback loops. In following discussion, We define phasers $\tilde{A}$ of RF frequency ω_{RF} ; $A(t) = \tilde{A}e^{i\omega_{RF}t}$ and phasers $\breve{A}$ of synchrotron frequency ω_s ; $A(t) = \breve{A}e^{i\omega_s t}$.

2 Response of Cavity Voltage to Current

The cavity voltage is driven by a beam current $i_b(t)$ and a generator current $i_g(t)$ produced by external RF source. We assume that these current has amplitude and phase oscillation of frequency ω and these drives the cavity voltage oscillation and the amplitude of these oscillation are small enough for us to linearize the system.

We divide the cavity voltage to the static part and oscillating part. The phases of them are

$$\tilde{V}_c(t) = \tilde{V}_{c0} + \Delta\tilde{V}_c(t), \quad \tilde{V}_{c0} = V_{c0}e^{i\phi_{c0}} \quad (1)$$

$$\tilde{V}_b(t) = \tilde{V}_{b0} + \Delta\tilde{V}_b(t), \quad \tilde{V}_{b0} = Z(\omega_{RF})\tilde{i}_{b0} \quad (2)$$

$$\tilde{V}_g(t) = \tilde{V}_{g0} + \Delta\tilde{V}_g(t), \quad \tilde{V}_{g0} = Z(\omega_{RF})\tilde{i}_{g0} \quad (3)$$

where $\tilde{V}_c = \tilde{V}_b + \tilde{V}_g$ is the phaser of the cavity voltage, $\tilde{V}_b$ is that induced by beam current and $\tilde{V}_g$ is that induced by generator current, and Z is the impedance of a cavity and $\tilde{i}_{b0} = i_{b0}e^{i\varphi_0}$ and $\tilde{i}_{g0} = i_{g0}e^{i\theta_0}$ are the static part of phasers of the beam current and the generator current, respectively.

The normalized variation of the phaser of the cavity voltage, with small and slow amplitude and phase oscillation of frequency $\omega \simeq \omega_s$, can be expressed as

$$\begin{aligned} \frac{\Delta\tilde{V}_c(t)}{\tilde{V}_{c0}} &= \frac{\tilde{V}_c(t) - \tilde{V}_{c0}}{\tilde{V}_{c0}} \\ &= \left[1 + \hat{k}\cos(\omega t + \alpha_k)\right] e^{i[\hat{\phi}\cos(\omega t + \alpha_\phi)]} - 1 \\ &\simeq \hat{k}\cos(\omega t + \alpha_k) + i\hat{\phi}\cos(\omega t + \alpha_\phi) \\ &= \frac{1}{2}\left(\breve{k} + i\breve{\phi}\right)e^{i\omega t} + \frac{1}{2}\left(\breve{k}^* + i\breve{\phi}^*\right)e^{-i\omega t} \quad (4) \end{aligned}$$

where $\breve{k} = \hat{k}e^{\alpha_k}$, $\breve{\phi} = \hat{\phi}e^{\alpha_\phi}$ and we assume $|\hat{k}|$, $|\hat{\phi}|$, $|\omega/\omega_{RF}| << 1$.

Cavity Voltage Oscillation by Beam Current

The one source of the oscillation of the cavity voltage is the phase oscillation of beam current produced by the synchrotron motion of bunches.

We will ignore the amplitude modulation of the beam current produced by unequal filling of bunches in a ring because its frequency spectrum is harmonics of revolution frequency and is much higher than synchrotron frequency and we will assume that the frequency response of the feed-back loop does not extend to such high frequency.

The beam current with the phase oscillation $\hat{\varphi}\cos(\omega t + \alpha_\varphi)$ is

$$\begin{aligned} i_b(t) &= i_{b0}e^{i[\varphi_0 + \hat{\varphi}\cos(\omega t + \alpha_\varphi)]}e^{i\omega_{RF}t} \\ &\simeq i_{b0}e^{i\varphi_0}\left[1 + i\hat{\varphi}\cos(\omega t + \alpha_\varphi)\right]e^{i\omega_{RF}t} \quad (5) \end{aligned}$$

where we used the assumption Eq. (??). The variation of the beam current $\Delta i_b(t) = i_b(t) - i_{b0}(t)$ presented by $\breve{\varphi} = \hat{\varphi}e^{i\alpha_\varphi}$ is

$$\Delta i_b(t) = \tilde{i}_{b0}\ i\frac{1}{2}\left[\breve{\varphi}e^{i\omega_{RF}^+ t} + \breve{\varphi}^* e^{i\omega_{RF}^- t}\right] \quad (6)$$

where $\omega_{RF}^+ = \omega_{RF} + \omega$ and $\omega_{RF}^- = \omega_{RF} - \omega$.

The cavity voltage induced by $\Delta i_b(t)$ is

$$\Delta V_b(t) = \tilde{i}_{b0}\ i\ \frac{1}{2}\left[\breve{\varphi}Z_+ e^{i\omega_{RF}^+ t} + \breve{\varphi}^* Z_- e^{i\omega_{RF}^- t}\right] \quad (7)$$

and its phaser is

$$\Delta\tilde{V}_b(t) = \frac{1}{2}\tilde{i}_{b0}\ i\ \left[\breve{\varphi}Z_+ e^{i\omega t} + \breve{\varphi}^* Z_- e^{-i\omega t}\right] \quad (8)$$

where $Z_+ = Z(\omega_{RF}^+)$ and $Z_- = Z(\omega_{RF}^-)$.

Cavity Voltage Oscillation by Generator Current

The other source of the cavity voltage oscillation is the generator current produced by external RF source. We set the generator current with phase and amplitude oscillation as

$$\begin{aligned} i_g(t) &= i_{g0}\left[1 + \hat{g}\cos(\omega t + \alpha_g)\right]e^{i\left[\omega_{RF}t + \theta_0 + \hat{\theta}\cos(\omega t + \alpha_\theta)\right]} \\ &\simeq i_{g0}(t)\left[1 + \hat{g}\cos(\omega t + \alpha_g) + i\hat{\theta}\cos(\omega t + \alpha_\theta)\right] \end{aligned}$$

. The variation $\Delta i_g(t) = i_g(t) - i_{g0}(t)$ presented by $\breve{g}$ and $\breve{\theta}$ is

$$\Delta i_g(t) = \frac{1}{2}\tilde{i}_{g0}\left[\left(\breve{g} + i\breve{\theta}\right)e^{i\omega_{RF}^+ t} + \left(\breve{g}^* + i\breve{\theta}^*\right)e^{i\omega_{RF}^- t}\right]. \quad (9)$$

Cavity Response to Beam and Generator Currnet

The cavity voltage induced by $\Delta i_g(t)$ is

$$\Delta V_g(t) = \tilde{i}_{g0}\left[\left(\breve{g} + i\breve{\theta}\right)Z_+ e^{i\omega_{RF}^+ t} + \left(\breve{g}^* + i\breve{\theta}^*\right)Z_- e^{i\omega_{RF}^- t}\right] \quad (10)$$

and its phaser is

$$\Delta \tilde{V}_g = \tilde{i}_{g0} \left[\left(\breve{g} + i\breve{\theta} \right) Z_+ e^{i\omega t} + \left(\breve{g}^* + i\breve{\theta}^* \right) Z_- e^{-i\omega t} \right]. \quad (11)$$

The cavity voltage amplitude variation induced by beam current and generator current, $\Delta\tilde{V}_c(t) = \Delta\tilde{V}_g(t) + \Delta\tilde{V}_b(t)$, is

$$\frac{\Delta\tilde{V}_c(t)}{\tilde{V}_{c0}} = \frac{1}{2\tilde{V}_{c0}} \left[\tilde{i}_{g0} \left(\breve{g} + i\breve{\theta} \right) + i\, \tilde{i}_{b0} \breve{\varphi} \right] Z_+ e^{i\omega t} + \frac{1}{2\tilde{V}_{c0}} \left[\tilde{i}_{g0} \left(\breve{g}^* + i\breve{\theta}^* \right) + i\, \tilde{i}_{b0} \breve{\varphi}^* \right] Z_- e^{-i\omega t} \quad (12)$$

Comparing above equation Eq. (12) and the equation Eq. (4), we have

$$\breve{k} + i\breve{\phi} = \frac{1}{\tilde{V}_{c0}} \left[\tilde{i}_{g0} \left(\breve{g} + i\breve{\theta} \right) + i\, \tilde{i}_{b0} \breve{\varphi} \right] Z_+ \quad (13)$$

$$\breve{k}^* + i\breve{\phi}^* = \frac{1}{\tilde{V}_{c0}} \left[\tilde{i}_{g0} \left(\breve{g}^* + i\breve{\theta}^* \right) + i\, \tilde{i}_{b0} \breve{\varphi}^* \right] Z_- (14)$$

Solving above equations for $\breve{k}$ and $\breve{\varphi}$, we have a response of a cavity voltage, $\breve{\mathbf{a}}$, to the phase oscillation of the beam current, $\breve{\mathbf{a}}_b$, and to the amplitude and phase oscillation of the generator current,$\breve{\mathbf{a}}_g$, in matrix form;

$$\breve{\mathbf{a}} = \breve{\mathbf{a}}_g + \breve{\mathbf{a}}_b, \quad \breve{\mathbf{a}}_g = R\breve{\mathbf{g}}, \quad \breve{\mathbf{a}}_b = C\breve{\boldsymbol{\varphi}} \quad (15)$$

where $\breve{\boldsymbol{\varphi}} = (1,1)^T \breve{\varphi}$,

$$\breve{\mathbf{a}} = \begin{pmatrix} \breve{k} \\ \breve{\phi} \end{pmatrix}, \breve{\mathbf{a}}_g = \begin{pmatrix} \breve{k}_g \\ \breve{\phi}_g \end{pmatrix}, \breve{\mathbf{a}}_b = \begin{pmatrix} \breve{k}_b \\ \breve{\phi}_b \end{pmatrix}, \breve{\mathbf{g}} = \begin{pmatrix} \breve{g} \\ \breve{\theta} \end{pmatrix} \quad (16)$$

and

$$R = \begin{pmatrix} r^+ & r^- \\ -r^- & r^+ \end{pmatrix}, C = \begin{pmatrix} c_- & 0 \\ 0 & c^+ \end{pmatrix} \quad (17)$$

$$r^\pm = \frac{i_{g0}}{2V_{c0}} \left(e^{i(\theta_0 - \phi_{c0})} Z_+ \pm e^{-i(\theta_0 - \phi_{c0})} Z_-^* \right) \quad (18)$$

$$c^\pm = \frac{i_{b0}}{2V_{c0}} \left(e^{-i\phi_{c0}} Z_+ \pm e^{i\phi_{c0}} Z_-^* \right). \quad (19)$$

3 Responce of Beam to Cavity Voltage

The discussion above is focused on voltages in a single cell of a cavity. Next, we will obtain the responce of the beam current to the acceleration voltage which was produced by many cavities in RF stations around the ring.

We assume that the number of RF stations is N_s and each station has one RF source and has two independent feed-back loops, one for the cavity voltage amplitude and one for phase. The parameters of cavities at each station are assumed to be the same and the feed-back loop controls the total voltage of them through controlling the RF-source. The parameters of each stations are specified by suffix i.

To treat multiple RF stations, we extend vectors, $\breve{\mathbf{a}}$, $\breve{\mathbf{a}}_b$ and $\breve{\mathbf{a}}_g$, and matrixes, C and R, as

$$\mathbf{v} = \begin{pmatrix} \breve{\mathbf{a}}_1 \\ \vdots \\ \breve{\mathbf{a}}_{N_s} \end{pmatrix}, M = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_{N_s} \end{pmatrix} \quad (20)$$

where $\breve{\mathbf{v}}_i$ and M_i are a vector and a matrix of i-th station.

We define energy shift $\delta = \frac{E-E_0}{E_0}$ and phase φ of bunches of the beam $\varphi = \omega_{RF}\tau$ where τ is the time advance relative to the reference particle. The equation for synchrotron oscillation of a bunch is

$$\frac{d\varphi}{dt} = -\omega_{RF}\, \eta\, \delta \quad (21)$$

$$\begin{aligned} \frac{d\delta}{dt} &= \sum_{s=1}^{N_s} \frac{eV_{c0,i}\,(1+k_i)}{T_0 E_0} \cos(-\varphi + \phi_i + \phi_{c0,i}) - \frac{U_0}{T_0E_0} - 2\frac{U_0}{T_0E_0}\delta \\ &\simeq \frac{1}{\omega_{RF}\eta} \frac{2}{\tau_s} \frac{d\varphi}{dt} + \left(\sum_{s=1}^{N_s} \frac{eV_{c0,i}}{T_0E_0} \sin\phi_{c0,i} \right) \varphi \\ &+ \sum_{s=1}^{N_s} \frac{eV_{c0,i}}{T_0E_0} \sin\phi_{c0,i} \left(\frac{k_i}{\tan\phi_{c0,i}} - \phi_i \right) \end{aligned} \quad (22)$$

where $k_i = \Re[\breve{k}_i e^{i\alpha_{k,i}}]$, $\phi_i = \Re[\breve{\phi}_i e^{i\alpha_{\phi,i}}]$, η is momentum compaction factor of the ring , U_0, E_0 and T_0 are the energy loss of a particle during one turn, the reference energy and the revolution period of the ring ,respectively. and we set $1/\tau_s = U_0/(T_0E_0)$ and

$$\sum_{s=1}^{N_s} \frac{eV_{c0,i}}{T_0E_0} \cos\phi_{c0,i} = \frac{U_0}{T_0E_0}. \quad (23)$$

where $\tilde{V}_{c0,i} = V_{c0,i}\, e^{i\phi_{c0,i}}$ is the phaser of cavity voltage of the i-th station.

From Eq. (21) and Eq. (22), we have

$$\frac{d^2\varphi}{dt^2} + \frac{2}{\tau_s}\frac{d\varphi}{dt} + {\omega_{s0}}^2 \varphi = -\sum_{s=1}^{N_s} {\omega_{s,i}}^2 \left(\frac{k_i}{\tan\phi_{c0,i}} - \phi_i \right) \quad (24)$$

where ${\omega_{s,i}}^2 = \frac{\omega_{RF}\eta}{T_0E_0} eV_{c0,i} \sin\phi_{c0,i}$ and ${\omega_{s0}}^2 = \sum_{s=1}^{N_s} {\omega_{s,i}}^2$.

From Eq. (15)-Eq. (29) and using phasers, Eq. (24) is

$$(\omega - \omega_+)(\omega - \omega_-)\breve{\varphi} = \sum_{s=1}^{N_s} {\omega_{s,i}}^2 \left(\frac{\breve{k}_{g,i}}{\tan\phi_{c0,i}} - \breve{\phi}_{g,i} \right) \quad (25)$$

where ω_+, ω_- are the solutions of the equation;

$$\left(\omega_\pm^2 - \frac{2}{\tau_s} i\omega_\pm - {\omega_{s0}}^2 \right) - \sum_{s=1}^{N_s} {\omega_{s,i}}^2 \left(\frac{c_i^-}{\tan\phi_{c0,i}} - c_i^+ \right) = 0. \quad (26)$$

Eq. (26) is the equation for the system without generator current variation ,$\breve{k}_{g,i} = \breve{\phi}_{g,i} = 0$,which is the case without feed-back loops and ω_+, ω_- show synchrotron frequency and growth rate of usual Robinson instability [1]. ω_- is related to ω_+ as $\omega_- = -\Re[\omega_+] + i\Im[\omega_-]$ because the original equation Eq. (24) is real.

Eq. (25) can be presented as matrix form,

$$\breve{\varphi} = \frac{I}{(\omega - \omega_+)(\omega - \omega_-)} T \breve{\mathbf{a}}_g. \qquad (27)$$

where T is the matrix of $rank(T) = 1$,

$$T = \begin{pmatrix} \frac{1}{\tan\phi_{c0,1}} & -1 & \cdots & \frac{1}{\tan\phi_{c0,N_s}} & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\tan\phi_{c0,1}} & -1 & \cdots & \frac{1}{\tan\phi_{c0,N_s}} & -1 \end{pmatrix}. \qquad (28)$$

Now we have the response of the phase oscillation of the beam current to the cavity voltage.

4 Generator Current driven by Feed-Back Loop

Combining Eq. (27) and Eq. (15), we have

$$\breve{\mathbf{a}} = \left[I + \frac{I}{(\omega - \omega_+)(\omega - \omega_-)} CT \right] R\breve{g}. \qquad (29)$$

The feed-back loop of i-th station detect $\breve{\mathbf{a}}_i$ and the difference from input signal, $\breve{\mathbf{a}}_{in,i}$, is filtered and amplified with a filter circuit of the impedance, $Z_{fk,i}$ and $Z_{f\phi,i}$, and amplifiers of the open gain, $G_{0k,i}$ and $G_{0\phi,i}$, for amplitude and phase loop, respectively, as shown in Fig. 1 . The resulting signal is the amplitude and the phase oscillation of the generator current,

$$\breve{\mathbf{g}} = G_0 Z_f (\breve{\mathbf{a}}_{in} - \breve{\mathbf{a}}) \qquad (30)$$

where G_0 and Z_f are diagonal matrix because feed-back loops are independent each other and they are

$$G_0 = \begin{pmatrix} G_{0,1} & & 0 \\ & \ddots & \\ 0 & & G_{0,N_s} \end{pmatrix}, \; G_{0,i} = \begin{pmatrix} G_{0k,i} & 0 \\ 0 & G_{0\phi,i} \end{pmatrix} \qquad (31)$$

$$Z_f = \begin{pmatrix} Z_{f,1} & & 0 \\ & \ddots & \\ 0 & & Z_{f,N_s} \end{pmatrix}, \; Z_{f,i} = \begin{pmatrix} Z_{fk,i} & 0 \\ 0 & Z_{f\phi,i} \end{pmatrix}. \qquad (32)$$

Solving Eq. (29) with Eq. (30) and Eq. (15) for $\breve{\mathbf{a}}$, we obtain

$$[(\omega - \omega_+)(\omega - \omega_-)(I + RG_0Z_f) + CTRG_0Z_f]\,\breve{\mathbf{a}} = [(\omega - \omega_+)(\omega - \omega_-) I + CT] RG_0Z_f \breve{\mathbf{a}}_{in}. \qquad (33)$$

5 Response to Input signal

From Eq. (33), we have

$$\breve{\mathbf{a}} = [(\omega - \omega_+)(\omega - \omega_-)(I + RG_0Z_f) + CTRG_0Z_f]^{-1} \times [(\omega - \omega_+)(\omega - \omega_-) I + CT] RG_0Z_f \breve{\mathbf{a}}_{in}. \qquad (34)$$

To get frequency response of the system, we set a vector $\breve{\mathbf{a}}_{in}$ and frequency ω.

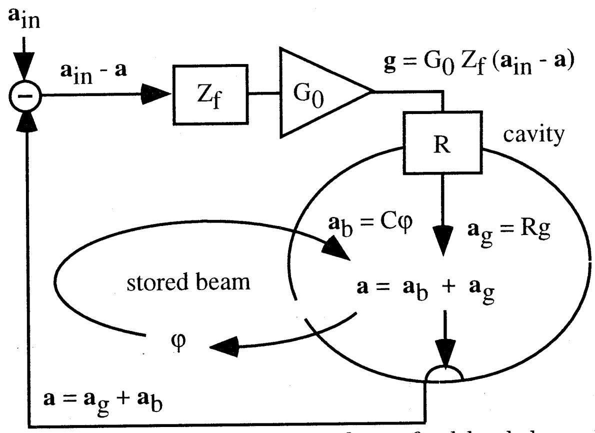

Fig. 1 Diagram of cavity voltage feed-back loop. ˘ for phasers are omitted here.

6 Growth Rate

The growth rate can be obtained by solving the equation Eq. (33) for ω with $\breve{\mathbf{a}}_{in} = 0$. For non-trivial solution, $\breve{\mathbf{a}} \neq 0$, we have an eigenvalue equation

$$\left| (\omega - \omega_+)(\omega - \omega_-) I + (I + RG_0Z_f)^{-1} CTRG_0Z_f \right| = 0 . \qquad (35)$$

Because $rank(T) = 1$ by the definition Eq. (28), there is only one non-zero eigenvalue for above equation. Thus the eigenvalue is easily obtained by taking trace of the matrix which is the sum of eigenvalues and is

$$(\omega - \omega_+)(\omega - \omega_-) = -Tr\left[(I + RG_0Z_f)^{-1} CTRG_0Z_f \right]. \qquad (36)$$

If we set $G_0 = 0$ or $Z_f = 0$, which is the case of no feed-back loop, the right-hand side of the Eq. (36) is zero and we have the solutions $\omega = \omega_+, \omega_-$ as expected.

When solving Eq. (36), we have to care that parameters in left-hand side of Eq. (36) such as C, R, G_0, Z_f may have frequency dependence.

7 Conclusion

The effect of the cavity voltage feed-back loop on the Robinson instability was analyzed and growth rate and synchrotron frequency shift is obtained. It shows that ,in actual machine which has slower synchrotron oscillation frequency and large beam loading like the SPring-8 storage ring, the frequency response of the feed-back loop must be slower enough to suppress gain at the synchrotron oscillation frequency to get stable operation at high current.

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References

[1] P. Wilson, SLAC-PUB-2884(1991),SLAC.