# Diffusion Instability in Noisy Beam-Beam Interaction

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### Abstract

Beam-beam instability in the presence of random fluctuations in opposite beam size is studied. Specific feature of noise beam-beam instability is that this instability can appear apart from excitation of nonlinear resonances. Diffusion coefficient of noise beam-beam instability exhibits quadratic dependence on the values of beam-beam parameter and noise amplitude and sharp dependence on normalized beam size. In the vicinity of low-order resonances beam size become larger, which results in much faster diffusion. Diffusion coefficient of instability as a function of beam-bean parameter, amplitude of noise and betatron tune value is presented.

### 1 Introduction

One of the main problems in ion-ion circular colliders is a small value of achievable beam-beam tune shift $\xi$=0.005. Physical reason for beam-beam limitation is usually attributed to the excitation of a set of nonlinear resonances due to a periodic nonlinear kick in linear system. Overlapping of nonlinear resonances is an universal mechanism of stochastic particle instability in nonlinear systems. Another mechanism of unstable particle motion is a diffusion, created by a noise [1-6]. This noise can exist, for example, due to mismatch of the beam with the channel. In this paper we study the noise which appears in an incoherent beam-beam interaction. As it is shown below, such a noise always induces diffusion instability if beam-beam kick is a nonlinear function of coordinate. Due to the diffusion character, noisy beam-beam instability does not have a threshold character and can exists under any value of beam-beam tune shift.

### 2 Noise Beam-Beam Instability

Noise beam-beam instability can be easily demonstrated via computer simulations. Let us consider for simplicity a one-dimensional model. Results of this study are valid for multi-dimensional problem as well. Particle motion is described in coordinates (x, p =$\beta$ dx/dz), where $\beta$ is a value of beta-function of collider. Between subsequent collisions particle experience linear matrix transformation with betatron angle $\theta = 2\pi Q$. Beam-beam interaction is treated as a thin lens with nonlinear beam-beam kick $\Delta p_n$:

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_n \\ p_n + \Delta p_n \end{pmatrix}. \qquad (1)$$

Beam-beam kick $\Delta p$ is expressed by a Gaussian function

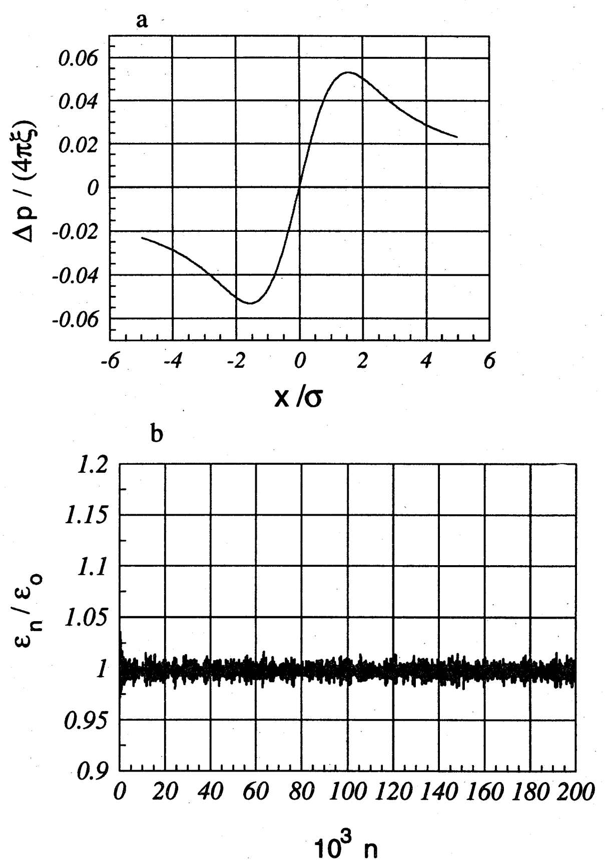


Fig. 1. Stable particle motion with the values of beam-beam tune shift $\xi$ =0.005 and betatron tune Q=3.168: a) beam-beam kick, b) rms beam emittance.

$$\Delta p = -4\pi\xi\, x \frac{1 - \exp(-x^2/2\sigma^2)}{(x^2/2\sigma^2)}, \qquad (2)$$

where x is a particle position and $\sigma$ is a standard deviation in opposite beam size. Parameter $\xi$ is a beam-beam parameter, which characterizes the strength of interaction:

$$\xi = \frac{N r_o \beta^*}{4\pi\sigma^2\gamma}, \qquad (3)$$

where N is a number of particles per bunch, $r_o = q^2/(4\pi\varepsilon_o mc^2)$ is a classical particle radius, and $\gamma$ is a particle energy.

Nonlinear kick (2) induces a set of nonlinear resonances, which are stable, until islands do not overlap each other. Isolated nonlinear resonance of the order p is described by Hamiltonian [7]:

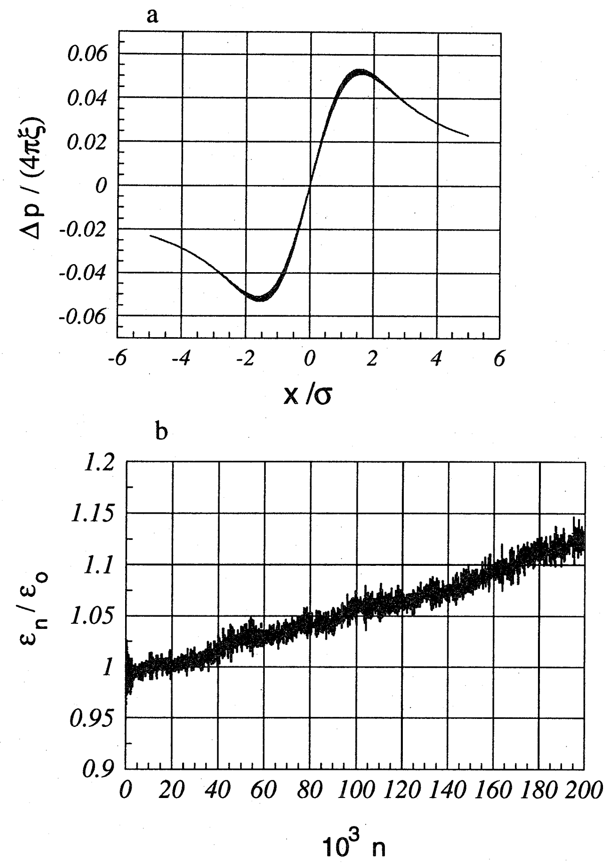


Fig.2. Beam-beam instability under 5% noise in the parameter $\sigma$ during particle interaction with $\xi = 0.005$ and Q=3.168: a) beam-beam kick, b) rms beam emittance growth.

$$H(I,\psi) = I\cdot\left(Q - \frac{m}{p}\right) + \xi\, U(I) + \xi\, V(I)\cos p\psi \ , \qquad (4)$$

where $(I, \psi)$ are action-angle variables, Q is a betatron tune, m and p are integers, U(I) is a detuning factor and V(I) is a resonance strength function.

In Fig. 1 an example of stable beam-beam interaction in the vicinity of the 6th order resonance is presented. In simulations the beam is presented as a collection of 3000 particles. Simulations are done for the value of betatron tune Q = 3.168, close to the 6th order resonance value 3.16666 (or 6Q=19). From the simulations it is clear that beam emittance is stable.

Another picture is observed if the parameter $\sigma$ of beam-beam kick (2) is a subject of noise. In the calculations, presented in Fig. 2, parameters of the process are chosen to be the same as those in Fig.1, but standard deviation $\sigma$ is changed from turn to turn according to the expression:

$$\sigma^{(n)} = \sigma^{(o)}\left(1 \pm \frac{u}{2}u_n\right) \ , \qquad (5)$$

where u is a noise amplitude and $u_n$ is a random noise function within the interval (0,1). It corresponds to the noise in the size of the opposite beam, which can exist due to small beam mismatch with the channel. The value of noise amplitude u=0.05 is chosen arbitrary to demonstrate appearance of diffusion-type instability in the presence of small random perturbation of beam-beam kick. As shown in Fig. 2, this noise destroys the stability. In contrast with Fig. 1, beam emittance expand with time.

Important feature of the noise regime is that this kind of instability can exist apart from the excitation of nonlinear resonances. Noise beam-beam instability appears if two conditions are met:

- beam-beam kick is a nonlinear function of coordinate;
- parameter of beam-beam kick (beam standard deviation $\sigma$) is a subject of noise.

## 3 Analytical Treatment of Effective Beam Emittance Growth

Let us provide analytical estimations of emittance growth under noise beam-beam interaction. Transfer matrix after n turns with arbitrary momentum kick at every turn, $\Delta p_i$ , is given by the expression [5]:

$$\begin{aligned} x_n &= a\cos(n\theta + \Psi) + \sum_{i=0}^{n-1} \Delta p_i \sin(n-i)\theta \ , \\ p_n &= -a\sin(n\theta + \Psi) + \sum_{i=0}^{n-1} \Delta p_i \cos(n-i)\theta \ , \end{aligned} \qquad (6)$$

where $\Psi$ is the initial phase of oscillations. Random beam-beam kick ($\Delta p_i$ ) can be expressed as a function of unperturbed trajectory. It gives an approximate treatment of the problem, valid for small values of perturbation.

Suppose, beam-beam kick is a nonlinear function of unperturbed trajectory:

$$\Delta p_i = 4\ \delta_i\, x_i^3 \ . \qquad (7)$$

We study evolution of the root-mean-square (rms) beam emittance $\varepsilon_n = 4\sqrt{<x_n^2> <p_n^2> - <x_n\, p_n>^2}$, where brackets mean averaging on initial phases of particles. Calculation of rms beam emittance gives:

$$\begin{aligned} \frac{\varepsilon_n^2}{16} &= \left[\frac{a^2}{2} + \frac{5}{2}a^6 \sum_{i=0}^{n-1} \delta_i^2\right]^2 - \\ &- \left[\frac{3a^4}{2}\sum_{i=0}^{n-1} \delta_i \sin(2n\theta - 2i\theta) - \frac{5a^6}{2}\sum_{i=0}^{n-1} \delta_i^2 \cos(2n\theta - 2i\theta)\right]^2 - \\ &- \left[\frac{3a^4}{2}\sum_{i=0}^{n-1} \delta_i \cos(2n\theta - 2i\theta) + \frac{5a^6}{2}\sum_{i=0}^{n-1} \delta_i^2 \sin(2n\theta - 2i\theta)\right]^2 . \end{aligned} \qquad (8)$$

Effective beam emittance growth is then as follows:

$$\frac{\varepsilon_n^2}{\varepsilon_0^2} = 1 + \delta^2 a^4\, n + \zeta(\delta^4) \ , \qquad (9)$$

$$\zeta(\delta^4) = 25\, a^8 \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} \delta_i^2 \delta_k^2 \left[1 - \cos(2i\theta - 2k\theta)\right] \ . \qquad (10)$$

Eq.(10) indicates emittance growth in presence of nonlinearity in beam-beam kick due to nonvanishing term $\delta^2$.

Comparison of eqs. (2), (7) and (9) gives an expression for a diffusion coefficient D in beam emittance growth under noise regime $\varepsilon_n/\varepsilon_o = \sqrt{1 + D\,n}$ :

$$D = \pi^2 (\xi u)^2 \left(\frac{a}{2\sigma_o}\right)^4 . \qquad (11)$$

Diffusion coefficient exhibits quadratic dependence on noise amplitude u and on value of beam-beam parameter $\xi$ as well as fourth-power dependence on normalized beam size ($a/2\sigma_o$). It indicates, that noise instability in beam-beam interaction appears under arbitrary small values of u and $\xi$.

## 4. Numerical Study of Diffusion Coefficient

In Figs. 3-5 results of numerical evaluation of diffusion coefficients are presented. Working point in simulations given in Figs. 3, 4 was chosen far enough from low-order nonlinear resonances, so the beam size was equal to that of the opposite beam $a=2\sigma_o$. As it follows from Figs. 3, 4, numerical values of the diffusion coefficient are close to the analytical estimation of Eq. (11).

Excitation of high order nonlinear resonances results in increase of beam size. Without noise in beam-beam kick the nonlinear resonances are stable. In the vicinity of nonlinear resonances, where beam size become larger, diffusion has to be faster, according to the expression (11). Diffusion coefficient as a function of betatron tune due to different beam size is summarized in Fig. 5. As seen, in the vicinity of low-order resonances diffusion goes much faster. The diffusion coefficient has a peak values near resonances of the 8-th order (Q=3.1285), 6-th order (Q=3.1666), 4-th order (Q=3.25) and 3rd order (Q=3.333). These values of betatron tune correspond to the largest values of diffusion coefficient.

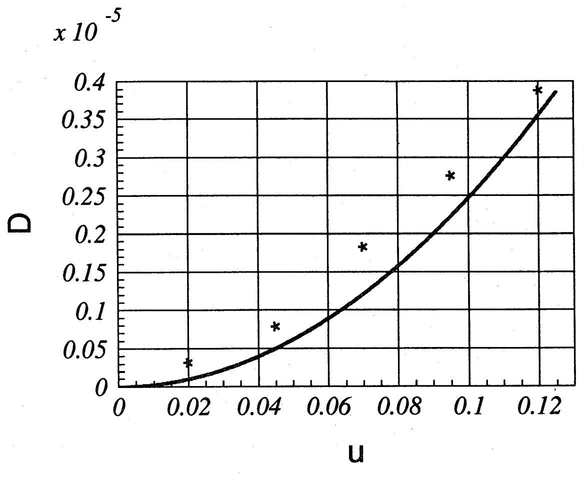


Fig. 3. Diffusion coefficient D as a function of noise amplitude u in parameter $\sigma$ for $\xi$=0.005: solid line - calculated from eq. (11), * - numerical values.

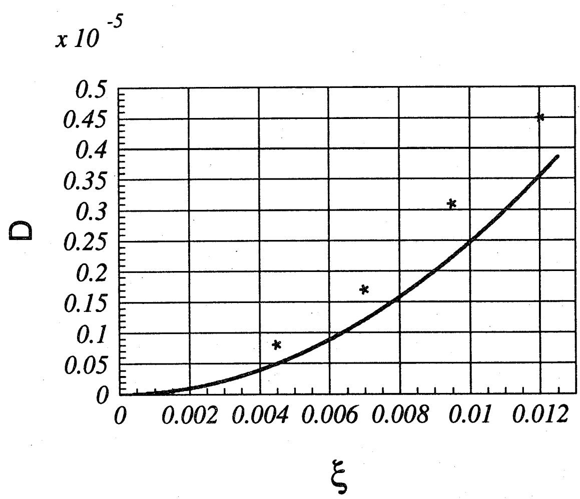


Fig. 4. Diffusion coefficient D as a function of beam-beam parameter $\xi$ for the value of noise amplitude u = 0.05: solid line - calculated from eq. (11), * - numerical values.

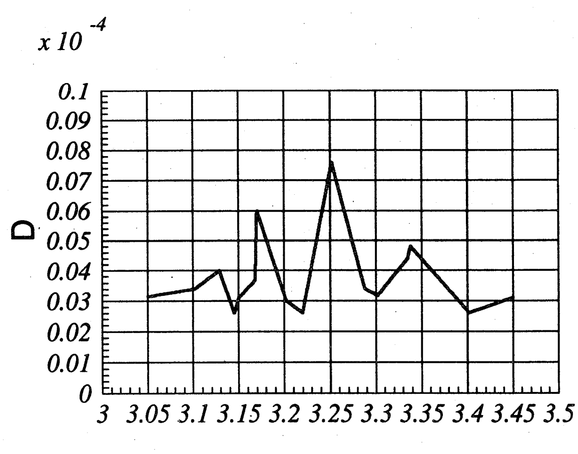


Fig. 5. Diffusion coefficient as a function of betatron tune in presence of noise with amplitude u=0.05.

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