Alternating Phase Focused Structures

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Abstract

New equations of motion for alternating phase focused (APF) structures are proposed. Formulas to evaluate the acceptances and current limits, which can be applied to any type of APF structures, are derived from the equations of motion. Coupled motion in an APF is also discussed in this paper.

Introduction

The APF accelerating method was first discovered about forty years ago and has been developed mainly by Russian researchers. There exist basically three types of APF structures, *i.e.* symmetric APF (SAPF), asymmetric APF (SAPF) and modified APF (MAPF)[1]. To study single particle motion in APF linacs, the two wave approximation (TWA) method was proposed[2]. The TWA is besed on travelling wave approach and is applicable to any type of APF structures. However, in the equations derived from the TWA method, synchronous phase alternation patterns cannot be explicitly introduced. The equations presented in this paper are obtained by using the stepping field approximation based on standing wave approach[3], and we can see explicitly the synchronous phase at each gap. Therefore, the present theory is more suitable for considering directly the various types of APF phase sequences and other design parameters at the gaps. In this paper, starting with the new equations of motion, we derive the formulas to estimate the acceptances and current carrying capacity of APF linacs. The effect of synchro-betatron coupling is also described.

Single particle equations of motion

Considering an axisymmetric drift tube structure whose periodic length is L_N, we can obtain, from the Maxwell's equations, the electromagnetic fields

$$\begin{cases} E_z(z,r,t) = \sum_{n=0}^{\infty} A_n I_0(k_n r)\cos\left(\frac{2n\pi z}{L_N}\right)\cos\omega t & (1a) \\ E_r(z,r,t) = \sum_{n=0}^{\infty} \frac{2n\pi A_n}{k_n L_N} I_1(k_n r)\sin\left(\frac{2n\pi z}{L_N}\right)\cos\omega t & (1b) \\ B_\theta(z,r,t) = -\sum_{n=0}^{\infty} \frac{A_n\omega}{k_n c^2} I_1(k_n r)\cos\left(\frac{2n\pi z}{L_N}\right)\sin\omega t & (1c) \end{cases}$$

where $I_n(x)$ is the modified Bessel function of the n-th kind, and $k_n^2=(2\pi/\lambda)^2[(n\lambda/L_N)^2-1]$. Here, an even-$\pi$ mode DTL has been assumed to make the following discussion simpler. Using Eqs.(1) and averaging the Lorentz force over single structure period $L_N=N\beta\lambda$, the transverse equation of motion is given by

$$\frac{dp_x}{d\tau} = -\frac{eqVT}{c\beta\gamma} I_1(\kappa x)\sin\phi \qquad (2)$$

where ϕ, q, V and T are, respectively, the synchronous phase, charge state of ion, the intergap voltage and transit-time factor at a gap, and $\kappa=2\pi/\beta\gamma\lambda$. The new independent parameter τ is defined as $\tau=\beta ct/L_N$. Applying the stepping field approximation presented in Ref.3, Eq.(2) can be rewritten for an APF structure as

$$\frac{dp_x}{d\tau} = -\frac{eq}{c\beta\gamma} I_1(\kappa x) \sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\sin(\Delta\phi+\phi_k^*) \qquad (3)$$

where

$$\Lambda_k(\tau) = 1+2\sum_{n=1}^{\infty} S_{nk}\cos[2n\pi(\tau-\tau_k)] \ , \quad S_{nk} = \frac{\sin(n\pi g_k/L_N)}{n\pi g_k/L_N} \ .$$

and

$$\tau_k(k\neq 1) = \frac{\beta\lambda}{2\pi L_N}\sum_{m=1}^{k-1}(\phi_{m+1}^* - \phi_m^* + \eta\pi) \quad with \quad \tau_1 = 0 \ .$$

In the above expression, the symbols with subscript k represent the values of the k-th cell in a focusing period, and g and η are, respectively, the gap length and accelerating mode number. Since the momentum p_x is related to the transverse coordinate as

$$\frac{dx}{d\tau} = \frac{L_N}{m_0 c\beta\gamma}\cdot p_x, \qquad (4)$$

we have from Eqs.(3) and (4)

$$\frac{d^2x}{d\tau^2} + \frac{2A}{\kappa} I_1(\kappa x) \sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\sin(\Delta\phi+\phi_k^*) = 0 \qquad (5)$$

where $A=\pi eqL_N/m_0c^2\beta^3\gamma^3\lambda$ and $\Delta\phi$ is the phase difference between synchronous and non-synchronous particles.
In similar way, we can obtain the longitudinal equation of motion as

$$\frac{d^2(\Delta\phi)}{d\tau^2} + 2A\sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)[I_0(\kappa x)\cos(\Delta\phi+\phi_k^*) - \cos\phi_k^*] = 0. \qquad (6)$$

Also for an odd-π mode DTL, we can obtain the same equations of motion.

Acceptances

When only the linear force terms in Eqs.(5) and (6) are taken into consideration, we have

$$\begin{cases} \frac{d^2x}{d\tau^2} + K_s(\tau)\, x = 0 & (7a) \\ \frac{d^2(\Delta\phi)}{d\tau^2} - 2K_s(\tau)(\Delta\phi) = 0 & (7b) \end{cases}$$

where

$$K_s(\tau) = B + \sum_{n=1}^{\infty} C_n \sin(2n\pi\tau + \theta_n)$$

with

$$B = \sum_{k=1}^{N} \Delta_k = \sum_{k=1}^{N} AV_k T_k \sin\phi_k^* \quad and$$

$$C_n = 2\left\{\left[\sum_{k=1}^{N} \Delta_k S_{nk}\sin(2n\pi\tau_k)\right]^2 + \left[\sum_{k=1}^{N} \Delta_k S_{nk}\cos(2n\pi\tau_k)\right]^2\right\}^{1/2} .$$

Neglecting the $C_n(n>1)$-terms in $K_s(\tau)$, the particle motion governed by Eqs.(7) has the stable region as shown in Fig.1.
Supposing the form of the solution of Eq.(7a) as

$$x(\tau) = \sqrt{\varepsilon_t \beta_t(\tau)}\cos(\sigma_{0t}\tau + \theta)$$

we obtain

$$\begin{cases} \beta_t(\tau) = \frac{L_N}{\sigma_{ot}}\left[1 + \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{C_n}{n^2}\sin(2n\pi\tau + \varphi'_n)\right] & (8a) \\ \sigma_{0t}^2 = B + \frac{1}{8\pi^2}\sum_{n=1}^{\infty}\left(\frac{C_n}{n}\right)^2 & (8b) \end{cases}$$

by using the smooth approximation.
When the bore radius of the drift tube is r_b, the transverse normalized acceptance is represented as

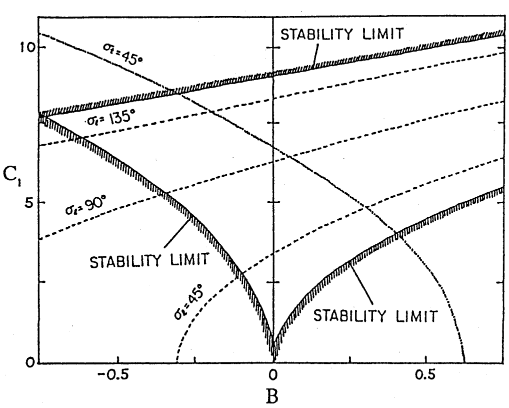

Fig.1. Stability region of accelerating particles in an APF linac

$$A_t = \beta\gamma \frac{r_b^2}{(\beta_t)_{max}} \qquad (9)$$

where $(\beta_t)_{max}$ is the maximum value of the betatron function evaluated from Eq.(8a).
We can also get from Eq.(7b) the smoothed longitudinal phase advance at zero current as

$$\sigma_\ell^2 = -2B + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left(\frac{C_n}{n}\right)^2 . \qquad (10)$$

By using the averaging method[3], the longitudinal potential function is approximately given by

$$U_{eff} = \frac{eq\beta\lambda}{2\pi L_N}\sum_{k=1}^{N} V_k T_k\left[\sin(\Delta\phi + \phi_k^s) - \Delta\phi\cos\phi_k^s\right] + \frac{1}{4m_0}\left(\frac{L_N}{2\pi\beta c}\right)^2 \sum_{n=1}^{\infty}\frac{u_n^2 + v_n^2}{n^2} + const. \qquad (11)$$

where

$$\begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \frac{2eq}{L_N}\sum_{k=1}^{N} V_k T_k S_{nk}\left[\cos(\Delta\phi + \phi_k^s) - \cos\phi_k^s\right]\times\begin{Bmatrix}\sin(2n\pi\tau_k) \\ \cos(2n\pi\tau_k)\end{Bmatrix}.$$

We can evaluate graphically the APF longitudinal acceptance from the width and the depth of the above effective potential well. Table.1 shows the acceptances of some phase sequences. The simulation results are obtained by Swenson[4] for 2π-mode operation. The agreement between the theoretically obtained results and the simulation results are quite good.

Space charge limit

In this section, we derive the current limit formulas by following the Wangler's method described in Ref.5, where a 3D ellipsoid having a uniform chage distribution is assumed. Introducing a space chage force term into Eq.(7a), we have

$$\frac{d^2 x}{d\tau^2} + \left[B + \Delta_{sc} + C_1\sin(2\pi\tau + \theta_1)\right] = 0 \qquad (12)$$

where

$$\Delta_{sc} = -\frac{3Z_0\, eqI\lambda L_N^2\,[1 - f(p)]}{8\pi m_0 c^2 r^2 b\beta^2\gamma^3},$$

I=beam current, Z_0=376.73Ω, r and b are the radius and longitudinal semi-aperture of the assumed ellipsoid respectively, and f(p) with p=b/r is the ellipsoid form factor[3]. Applying the Wangler's approach to Eq.(12), the transverse current limit formula is given by

$$I_t = \frac{8\pi}{3Z_0}\mu_t\frac{m_0c^2}{eq}\frac{\beta^2\gamma^3 \langle r\rangle^2 \langle b\rangle \sigma_{0t}^2}{L_N^2\,\lambda[1 - f(p)]} \qquad (13)$$

where μ_t is the transverse space charge parameter,

$$\langle r\rangle^2 = \frac{1 - C_1/4\pi^2}{1 + C_1/4\pi^2} r_b^2 \quad and \quad \langle b\rangle^2 = \frac{1 - C_1/2\pi^2}{1 + C_1/2\pi^2} b_{max}^2 .$$

Similarly, we obtain the longitudinal current limit as

$$I_\ell = \frac{4\pi}{3Z_0}\mu_\ell\frac{m_0c^2}{eq}\frac{\beta^2\gamma^3 \langle r\rangle^2 \langle b\rangle \sigma_{0\ell}^2}{L_N^2\,\lambda f(p)}. \qquad (14)$$

As an example, let us put λ=10m, N=2, r_b=4mm, the charge-to-mass ratio=1/20, and $\mu_t = \mu_\ell = 0.84$. In this case, the current limit is about 2mA.

Coupled motion[6]

Taking the second order coupling terms in Eqs.(5) and (6) into consideration, we obtain

$$\begin{cases} \dfrac{d^2 x}{d\tau^2} + K_s(\tau)\,x = -K_c(\tau)(\Delta\phi)\,x & (15a) \\ \dfrac{d^2(\Delta\phi)}{d\tau^2} - 2K_s(\tau)(\Delta\phi) = K_c(\tau)\left[(\Delta\phi)^2 - \dfrac{(\kappa x)^2}{2}\right] & (15b) \end{cases}$$

where

$$K_c(\tau) = B' + \sum_{n=1}^{\infty} C'_n \sin(2n\pi\tau + \theta'_n) .$$

Here, B' and C_n' are given by replacing Δ_k in B and C_n by $\Delta_k' = AV_kT_k\cos\phi_k^s$. Assuming that

$$x = \chi\cdot(1 + q_t) \quad and \quad \Delta\phi = \varphi\cdot(1 + q_\ell),$$

Eqs.(15) are smoothed to give

$$\begin{cases} \dfrac{d^2\chi}{d\tau^2} + \sigma_{0t}^2\,\chi = -\sigma_a^2\,\chi\varphi & (16a) \\ \dfrac{d^2\varphi}{d\tau^2} + \sigma_{0\ell}^2\,\varphi = \sigma_b^2\varphi^2 - \dfrac{1}{2}\kappa^2\sigma_c^2\chi^2 & (16b) \end{cases}$$

where

$$\begin{cases} \sigma_a^2 = \left[1 - \dfrac{1}{16\pi^4}\displaystyle\sum_{n=1}^{\infty}\left(\frac{C_n}{n^2}\right)^2\right]B' - \dfrac{1}{8\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{C_nC'_n}{n^2} \\ \sigma_b^2 = \left[1 + \dfrac{1}{8\pi^4}\displaystyle\sum_{n=1}^{\infty}\left(\frac{C_n}{n^2}\right)^2\right]B' - \dfrac{1}{2\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{C_nC'_n}{n^2} \\ \sigma_c^2 = \left[1 + \dfrac{1}{32\pi^4}\displaystyle\sum_{n=1}^{\infty}\left(\frac{C_n}{n^2}\right)^2\right]B' + \dfrac{1}{4\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{C_nC'_n}{n^2} . \end{cases}$$

N	Phase sequence (deg)	E_0T (MV/m)	Normalized transverse acceptance A_t (cm mrad)		Total phase acceptance $\Delta\phi_{acc}$(deg)	
			1	2	1	2
2	−60, 60	~14.0	3.23	3.14	70	62
2	−65, 55	~14.0	2.58	2.47	86	76
2	−70, 70	~12.0	2.93	2.92	74	76
3	−90, 30, 30	~8.0	1.83	1.93	58	52
3	−90, 40, 40	~10.3	3.60	3.54	52	50
4	−90, 0, 90, 0	~7.0	1.71	1.82	60	53
4	−60, −60, 60, 60	~4.5	1.45	1.57	50	50
4	−70, −70, 60, 60	~4.5	1.38	1.23	70	73
5	−90, −30, 60, 60, −30	~3.5	0.72	0.75	60	60
5	−90, −90, 30, 90, 30	~3.0	1.18	1.23	70	73
6	−90, −90, 0, 60, 60, 0	~2.8	0.84	0.85	65	76
6	−90, −90, 0, 70, 70, 0	~2.7	0.96	1.06	70	71
6	−90, −90, 0, 90, 90, 0	~2.6	1.13	1.19	60	70
7	−90, −90, 0, 40, 70, 40, 0	~2.4	1.11	1.25	45	49
8	−90, −90, −30, 30, 60, 60, 30, −30	~1.8	0.62	0.76	62	65
8	−90, −90, −30, 30, 90, 90, 30, −30	~1.8	0.81	0.98	70	64

Table.1. The normalized transverse acceptances and the total phase acceptances of proton APF linac, whose operating frequency and the beam energy are 400MHz and 1MeV respectively, are shown in this table. Each transverse acceptance is evaluated in the case where the linac bore radius is 1cm. E_0 is the averaged accelerating field used in the theoretical calculation and defined as $E_0 = V/\beta\lambda$ here. The values in columns 1 and 2 correspond to the simulation results presented in Ref.4 and the theoretically obtained results respectively.

It is easy to show that the coupling terms in Eqs.(16) yield a resonance given by the condition

$$2\sigma_{0t} - \sigma_{0l} = 0 .$$

This condition is equivalent to B=0 and, accordingly, symmetric phase sequence must be avoided to eliminate the lowest order resonance. Some straightforward algebra lead to the unstable width around this resonance as

$$1-\frac{1}{4}\left(\frac{\sigma_a}{\sigma_{0t}}\right)^2|\varphi_1| < \frac{\sigma_{0l}}{2\sigma_{0t}} < 1+\frac{1}{4}\left(\frac{\sigma_a}{\sigma_{0t}}\right)^2|\varphi_1|$$

where φ_1 is approximately equal to the half-width of the longitudinal phase acceptance evaluated from the effective potential Eq.(11). Therefore, noting the fact that σ_{0l} is usually taken larger than $2\sigma_{0t}$ to make the longitudinal acceptance as large as possible, the following condition must be satisfied:

$$\sigma_{0l} > 2\sigma_{0t}\cdot\left[1+\left(\frac{\sigma_a}{2\sigma_{0t}}\right)^2|\varphi_1|\right] \qquad (17)$$

Adiabatic damping law

In this section, we put $\gamma \sim 1$ because an APF structure is suitable for the low-β region, and the 2π-mode acceleration where $L_N = N\beta\lambda$ is considered here. The Hamiltonian for the longitudinal motion is written as

$$H = -\frac{\pi L_N}{m_0 c^2 \beta^2}(\Delta W)^2 + U(\Delta\phi)$$

where ΔW is the energy difference between synchronous and non-synchronous particles.
Assuming $\Delta\phi \ll 1$ and $U = U_{eff}$, the above Hamiltonian is reduced approximately to

$$H = -\frac{\pi N}{m_0 c^2 \beta^2}(\Delta W)^2 - \frac{m_0 c^2 \beta^2}{4\pi^2 N^2}\Lambda(\Delta\phi)^2 \qquad (18)$$

where

$$\Lambda = -B + \sum_{n=1}^{\infty}\left(\frac{C_n}{2n\pi}\right)^2 .$$

For the motion on the stability region shown in Fig.1, $\Lambda>0$. Using Eq.(18),

$$\frac{\Delta W}{\Delta\phi} = \frac{m_0 c^2 \beta^2}{2\pi N}\left(\frac{\Lambda}{\pi N}\right)^{1/2}$$

Let us indicate each parameter's value at the entrance and at the exit of an accelerator with the subscripts 'in' and 'out' respectively. From the Liouville's theorem, $\Delta\phi_{in}\Delta W_{in} = \Delta\phi_{out}\Delta W_{out}$. Therefore, we obtain

$$\frac{\Delta\phi_{out}}{\Delta\phi_{in}} = \frac{\beta_{in}}{\beta_{out}}\left(\frac{\Lambda_{in}}{\Lambda_{out}}\right)^{1/4} \qquad (19a)$$

$$\frac{\Delta W_{out}}{\Delta W_{in}} = \frac{\beta_{out}}{\beta_{in}}\left(\frac{\Lambda_{out}}{\Lambda_{in}}\right)^{1/4} \qquad (19b)$$

Because Eqs.(19) are based on Eq.(6) in which the change of β through a focusing period is supposed to be neglected, we must note that Eqs.(19) may lead to invalid results as the value of N or the energy gain at a gap becomes larger.

Determination of APF operating point

Let us consider a method to determine an APF operating point on the stability diagram. Since rf-frequency, intervane voltage, initial beam energy and the other fundamental parameters except for the phase sequence are chosen first, we can calculate the transverse acceptance A_t accordingly by using Eq.(9). In this case, we have the following relation between B and C_1:

$$\frac{L_N A_t}{\beta\gamma r_b^2} = \frac{\left(B + \frac{C_1^2}{8\pi^2}\right)^{1/2}}{\left(1 + \frac{C_1}{4\pi^2}\right)^2} \qquad (20)$$

If we take a same phase sequence throughout an APF structure, the longitudinal acceptance becomes smaller with increasing beam energy, so, in a usual APF design, the focusing period number N is taken larger in higher-β region while the transverse acceptance becomes smaller. Therefore, the bore radius r_b should be made gradually larger toward the high energy end to compensate the transverse acceptance decrease due to the N-value change, so that the r.h.s. of Eq.(20) is kept almost constant for constant A_t. Then, we must choose the desired value of the longitudinal phase acceptance $\Delta\phi_{acc}$. The B- and C_1-dependence in Eq.(11) is not obvious, so we cannot have such a clear formula as Eq.(20). However, according to a numerical calculation, $\Delta\phi_{acc}$ has a specific value corresponding to the values of B and C_1, and we can obtain a phase acceptance contour as shown in Fig.2. Therefore, when we select a value of $\Delta\phi_{acc}$, we can determine the parameters B and C_1 from the intersecting point of the contour and the curve obtained by Eq.(20). The representations of B and C_1 have already been given as

$$\frac{B}{A} = \sum_{k=1}^{\infty} P_k \qquad (21a)$$

$$\left(\frac{C_1}{2A}\right)^2 = \left[\sum_{k=1}^{\infty} P_k S_{1k}\sin(2\pi\tau_k)\right]^2 + \left[\sum_{k=1}^{\infty} P_k S_{1k}\cos(2\pi\tau_k)\right]^2 \qquad (21b)$$

where $P_k = V_k T_k \sin\phi_k$.
If N=2, we can determine uniquely a phase sequence corresponding to the specific values of B and C_1 by using Eqs.(21), noting the condition given by Eq.(17). In the case where N>2, we can also obtain a sequence with the additional consideration of maximum energy gain. Furthermore, when the beam current is large, the equipartitioning and matching conditions, *i.e.*

$$\frac{\varepsilon_l}{\varepsilon_t} = \frac{\sigma_t}{\sigma_l} = \frac{<b>}{<r>}$$

$$with \quad \varepsilon_t = \frac{\sigma_t <r>^2}{L_N} \quad and \quad \varepsilon_l = \frac{\sigma_l <b>^2}{L_N},$$

should also be considered for minimum emittance growth.

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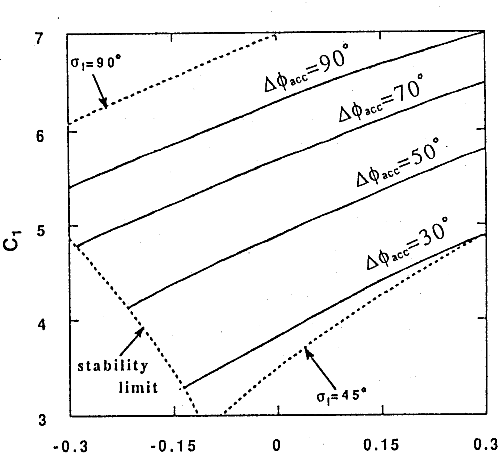

Fig.2. Phase acceptance contour example. These contours correspond to the N=2 APF linac for protons whose operating frequency, beam energy and averaged field amplitude are 400MHz, 100keV and 10MV/m respectively.